Pergamon

Appl. Math. Lett. Vol. 8, No. 6, pp. 15–17, 1995

0893-9659/95 $9.50 + 0.00

0893-9659(95)00078-X

Limit Cycles of Generalized Liénard Equations

S. LYNCH
Department of Mathematics and Physics, Manchester Metropolitan University
Manchester M15 6BH, England

(Received May 1995; accepted June 1995)

Abstract—The generalized Liénard equations of the form:

$$\dot{x} = h(y) - F(x), \qquad \dot{y} = -g(x),$$

where F, g, and h are polynomials, are examined. It has been found that the results given by Blows, Lloyd and Lynch [1–5] for Liénard equations hold also for the generalized systems. A new result is also presented within this article.

Keywords—Liénard equations, Limit cycles, Bifurcations.

In recent years, there has been intensive study on the bifurcations of limit cycles in planar polynomial vector fields. Systems of physical interest have been studied using this theory since the early investigations of H. Poincaré. There are two basic types of bifurcations: global bifurcations and local bifurcations. In a survey article, Blows and Perko [6] discuss global bifurcations from centres and separatrix cycles of planar analytic systems. Some bifurcations of limit cycles at infinity are studied for polynomial vector fields with no singular points at infinity by Blows and Rousseau [7]. The study of Liénard equations of the form:

$$\dot{x} = y - F(x), \qquad \dot{y} = -g(x), \tag{1}$$

has proved very useful in the investigation of limit cycles of planar systems. The theory appears extensively in the literature and has proved particularly useful when showing the existence, uniqueness, and hyberbolicity of a limit cycle. During the last twenty years, many papers have appeared when considering numbers of limit cycles of system (1). By considering the system with $g(x) = x$, Lins *et al.* [8] showed that if $M = 2m + 1$ or $2m + 2$ and $p \le m$, then there are polynomials F of degree M such that (1) has exactly p limit cycles. They also conjectured that there can be no more than m limit cycles. They showed that the conjecture is true if the degree of F is three, and Zeng [9] has proven that it holds in several cases when the degree of F is four. When the degree of F is five, Rychkov [10] showed that there are no more than two limit cycles if F is odd. When the degrees of F and g are both quadratic, Dumortier and Roussarie [11] have shown that there is at most one limit cycle. They also adduce that there is strong evidence to suggest that there is no more than one limit cycle when the degree of F is two and the degree of g is three. There have been relatively few results for the generalized Liénard system; see Zeng *et al.* [12], for example. Many mathematicians have investigated the algebraicity of limit cycles, most of them either Russian or Chinese; Odani [13] is the first to give a concrete example of a nonalgebraic limit cycle.

The work presented here is concerned with local bifurcations; bounds are obtained on the maximum possible number of limit cycles of small amplitude for the generalized Liénard equations.

Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX

These are equations of the form:

$$\dot{x} = h(y) - F(x), \qquad \dot{y} = -g(x), \tag{2}$$

where $h(y) = y + c_2y^2 + c_3y^3 + \cdots + c_Ly^L$, $F(x) = a_1x + a_2x^2 + a_3x^3 + \cdots + a_Mx^M$, and $g(x) = x + b_2x^2 + b_3x^3 + \cdots + b_Nx^N$; L, M, and N are natural numbers. Many of the recent results on polynomial systems have been obtained by bifurcating small-amplitude limit cycles, from critical points, by perturbations of the coefficients arising in the differential equations. This is particularly true for Liénard systems; see [1–5,14], for example. All of these results are local, since the limit cycles appear in a small neighbourhood around the critical point. The author has recently shown that the results given in [1–5] for system (1) also hold for the generalized system (2) when the degree of h is greater than one. The methods of proof, for 1–6 below, are similar to those appearing in [1] and [2], but more terms are involved.

Let $\hat{H}$ be the maximum number of small-amplitude limit cycles that can bifurcate from the origin, and let ∂ denote the degree of a polynomial. Suppose that $\partial h \geq 1$. The following results have been proven for system (2):

1. If $\partial g = 1$ and $\partial F = 2m+1$ or $2m+2$, then $\hat{H} = m$.
2. If g is odd and $\partial F = 2m+1$ or $2m+2$, then $\hat{H} = m$.
3. If $\partial F = 2$ and $\partial g = 2n$ or $2n+1$, then $\hat{H} = n$.
4. If F is odd, $\partial F = 2m+1$, then $\hat{H} = m$, whatever g is.
5. If F is even, $\partial F = 2m+2$ and $\partial g = 2$, then $\hat{H} = m$.
6. If F is even, $\partial F = 2m+2$ and $\partial g = 2n+2$ or $2n+3$, then $\hat{H} = \max(m,n)$.

The author has recently obtained the following result using similar arguments to those appearing in [2]. This is a new result:

7. If $\partial F = 3$, $g(x) = x + g_e(x)$, where g_e is even and $\partial g = 2n$, then $\hat{H} = n$.

Complementing these results is the calculation of $\hat{H}$ when F and g are of specified degrees and $\partial h \geq 1$. The results are presented in Table 1, which has been updated since appearing in [3–5].

Table 1. The values of $\hat{H}$ for varying degrees of F and g when $\partial h \geq 1$.

Degree of F

14	↑												
13	6	8											
12	5	7											
11	5	7											
10	4	6	8										
9	4	5	8										
8	3	5	6										
7	3	4	6	8									
6	2	3	4	6	8								
5	2	3	4	4	6	8							
4	1	2	2	4	4	6	6	8	8				
3	1	1	2	3	3	4	5	5	6	7	7	8	
2	0	1	1	2	2	3	3	4	4	5	5	6	→
	1	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**	**12**	**13**

Degree of g

The question of the relationship between local and global results for these systems is still open and of considerable interest.

REFERENCES

1. N.G. Lloyd and T.R. Blows, The number of small-amplitude limit cycles of Liénard equations, *Math. Proc. Camb. Philos. Soc.* **95**, 359–366 (1984).
2. N.G. Lloyd and S. Lynch, Small-amplitude limit cycles of certain Liénard systems, *Proc. Roy. Soc. Lond. Ser. A* **418**, 199–208 (1988).
3. S. Lynch, Small-amplitude limit cycles of Liénard equations, *Calcolo* **127** (1–2) (1990).
4. S. Lynch, More results on the bifurcation of limit cycles for systems of Liénard type, *J. Egypt. Math. Soc.* **2**, 75–87 (1994).
5. S. Lynch, Small-amplitude limit cycles of the generalized mixed Rayleigh-Liénard oscillator, *Journal of Sound and Vibration* **178** (5), 615–620 (1994).
6. T.R. Blows and L.M. Perko, Bifurcation of limit cycles from centres and separatrix cycles of planar analytic systems, *SIAM Review* **36** (3), 341–376 (1994).
7. T.R. Blows and C. Rousseau, Bifurcation at infinity in polynomial vector fields, *J. Diff. Eqn.* **104**, 215–242 (1993).
8. A. Lins, W. de Melo and C. Pugh, On Liénards equation with linear damping, *Lecture Notes in Mathematics*, No. 597, (Edited by J. Palis and M. do Carmo), pp. 335–357, Springer-Verlag, Berlin, (1977).
9. X. Zeng, Remarks on the uniqueness of limit cycles, *Kexue Tongbao* **28**, 452–455 (1983).
10. G.S. Rychkov, The maximum number of limit cycles of the system $\dot{x} = y - a_1x^3 - a_2x^5$, $\dot{y} = -x$ is two, *Differentsial'nye Uravneniya* **11**, 380–391 (1973).
11. F. Dumortier and R. Roussarie, Cubic Liénard equations with linear damping, *Nonlinearity* **3**, 1015–1039 (1990).
12. X. Zeng, Z.-F. Zhang and S. Gao, On the uniqueness of the limit cycle of the generalized Liénard equation, *Bull. Lond. Math. Soc.* **26**, 213–247 (1994).
13. K. Odani, The limit cycle of the van der Pol equation is not algebraic, *J. Diff. Eqn.* **115**, 146–152 (1995).
14. A.M. Urbina, G. León de la Barra, M. León de la Barra and M. Cañas, Limit cycles of Liénard equations with nonlinear damping, *Canad. Math. Bull.* **36** (2), 251–256 (1993).